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On the riemannian metrics in $\mathbb{R}^n$ which admit all hyperplanes as minimal hypersurfaces

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Abstract

Inspired by a result of Bekkar (1991), Robert Lutz raised the following problem: determine the riemannian metrics in domains of $\mathbb{R}^n$ which admit all hyperplanes as minimal hypersurfaces. We solve the problem giving a formula which expresses its solutions in terms of the non-degenerate quadratic first integrals of the geodesic motion in the euclidean space (second-order Killing tensor fields). Then, we prove that for $n = 3$ the non-flat polynomial solutions of the problem are the left invariant riemannian metrics on the Heisenberg group.

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1. Introduction

A couple (D, g), where D is a connected open neighbourhood of $\mathbb{R}^n$ and g a riemannian metric defined in D, will be said to satisfy the property $\mathcal{L}_n$ if all the intersections $p \cap D$ of D with the hyperplanes p of $\mathbb{R}^n$ are minimal hypersurfaces of the riemannian neighbourhood (D, g).

The couple $(\mathbb{R}^n, g_{\mathrm{E}})$ where g_{E} indicates the canonical euclidean riemannian structure of $\mathbb{R}^n$ satisfies the property $\mathcal{L}_n$.

Left invariant riemannian structures g_{H} on the Heisenberg group H_1 (the model in view in this note is $\mathbb{R}^3$ endowed with the group law $\mu_k : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, $k \in \mathbb{R}^*$ defined by

$$\mu_k((x, y, z), (x', y', z')) = (x + x', y + y', z + z' + \tfrac{1}{2}kxy' - \tfrac{1}{2}kx'y))$$

provide us with examples $(\mathbb{R}^3, g_{\mathrm{H}}) \in \mathcal{L}_3$, see [1].

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Inspired by this example, R. Lutz raised the following problem: characterize all the couples $(D, g) \in \mathcal{L}_n$. In dimension $n = 3$ and for axially symmetric g, the problem was solved in [2].

In what follows we first establish the P.D.E. system, denoted $\mathcal{E}_n$, satisfied by the components g_{ij} of the tensor g in order that $(D, g) \in \mathcal{L}_n$. Then, we integrate the system $\mathcal{E}_n$ and express its solutions in terms of non-degenerate second-order Killing tensor fields of $(\mathbb{R}^n, g_E)$. Finally, we characterize the riemannian structures $(\mathbb{R}^3, g_H)$ as the only non-flat polynomial solutions of $\mathcal{E}_3$.

2. The system $\mathcal{E}_n$

If one denotes $X^1, X^2, ..., X^n$ the cartesian coordinates of the affine space $\mathbb{R}^n$, the equation of a hyperplane p which does not contain the origin $O \in \mathbb{R}^n$ writes

$$p_1 X^1 + p_2 X^2 + \cdots + p_n X^n = 1.$$

The parameters $(p_1, p_2, ..., p_n)$ will be considered as local coordinates on the variety of all hyperplanes p of $\mathbb{R}^n$. A riemannian structure g in $\mathbb{R}^n$ is defined by a positive definite quadratic form in $\mathrm{d}X^1, \mathrm{d}X^2, ..., \mathrm{d}X^n$ say

$$\mathrm{d}s^2 = \sum_{i,j=1}^{n} g_{ij}(X^1, ..., X^n)\, \mathrm{d}X^i\, \mathrm{d}X^j.$$

At a point $x = (x^1, x^2, ..., x^n) \in \mathbb{R}^n$ which belongs to the hyperplane p $(p_1 x^1 + \cdots + p_n x^n = 1)$ the equation of the tangent space at x to p, denoted $(Tp)_x$,writes

$$(p_1\, \mathrm{d}X^1 + p_2\, \mathrm{d}X^2 + \cdots + p_n\, \mathrm{d}X^n)_x = 0.$$

The parameters p_i, $i = 1, 2, ..., n$, represent the covariant components of a vector $\boldsymbol{p}(x)$, normal to the hyperplane p at x, the norm of this vector being

$$\|\boldsymbol{p}(x)\| = \left(\sum_{r,s} g^{rs}(x) p_r p_s \right)^{1/2}.$$

Therefore $n : x \in p \longrightarrow n(x) = \boldsymbol{p}(x)/\|\boldsymbol{p}(x)\|$ is a unit vector field normal to p; it will be used to introduce the second fundamental form of p, denoted Φ_p. Its covariant components in $(\mathbb{R}^n, g)$ are

$$n_i(x) = \frac{p_i}{\|\boldsymbol{p}(x)\|}, \quad i = 1, 2, ..., n.$$

The second fundamental form $\Phi_{p,x}$ at x is the restriction to $(Tp)_x$ of the quadratic form

$$-g(\nabla n(x), \mathrm{d}X) = -\sum_{i=1}^{n} \left(\mathrm{d}n_i(x) - \sum_{r,s} \{{}_s{}^r{}_i\}_x n_r(x)\, \mathrm{d}X^s \right) \mathrm{d}X^i,$$

where ∇ denotes the Levi–Civita connection of g and $\{_s{}^r{}_i\}_x$ are the values at x of the Christoffel symbols of the second kind of g. Clearly

$$\Phi_{p,x} = \sum_{r,s,i} \{_s{}^r{}_i\}_x \frac{p_r}{\|\boldsymbol{p}(x)\|} \mathrm{d}X^s \, \mathrm{d}X^i \Big|_{\sum_u p_u \, \mathrm{d}X^u = 0}.$$

Consider now the quadratic form on $(T\mathbb{R}^n)_x$, $x \in p$,

$$\Phi'_{p,x} = \sum_{r,s,i} \{_s{}^r{}_i\}_x \frac{p_r}{\|\boldsymbol{p}(x)\|} \left(\mathrm{d}X^s - \frac{p^s(x)}{\|\boldsymbol{p}(x)\|^2} \sum_u p_u \, \mathrm{d}X^u \right) \times \left(\mathrm{d}X^i - \frac{p^i(x)}{\|\boldsymbol{p}(x)\|^2} \sum_v p_v \, \mathrm{d}X^v \right)$$

where $p^i(x) = \sum_r g^{ir}(x) p_r$ are the contravariant components of the vector $\boldsymbol{p}(x)$. One has

$$\Phi_{p,x} = \Phi'_{p,x} \big|_{(Tp)_x}$$

and the restriction of $\Phi'_{p,x}$ to the normal space to p at x vanishes identically as

$$\left(\mathrm{d}X^i - \frac{p^i(x)}{\|\boldsymbol{p}(x)\|^2} \sum_v p_v g \, \mathrm{d}X^v \right) (\boldsymbol{p}(x)) = p^i(x) - \frac{p^i(x)}{\|\boldsymbol{p}(x)\|^2} \sum_v p_v p^v(x) = 0.$$

Therefore, the trace of $\Phi'_{p,x}$ with respect to g_x in $(T\mathbb{R}^n)_x$ represents nothing but the mean curvature $H_{p,x}$ of p at $x \in p$ and one has

$$H_{p,x} = \sum_{u,v,r,s,t} g_x^{uv} \{_s{}^r{}_t\}_x \frac{p_r}{\|\boldsymbol{p}(x)\|} \left(\delta_u^s - \frac{p^s(x)}{\|\boldsymbol{p}(x)\|^2} p_u \right) \left(\delta_v^t - \frac{p^t(x)}{\|\boldsymbol{p}(x)\|^2} p_v \right)$$

which writes also

$$\|\boldsymbol{p}(x)\|^3 H_{p,x} = \sum_{u,v,r} g_x^{uv} \{_u{}^r{}_v\}_x p_r \|\boldsymbol{p}(x)\|^2 - \sum_{u,v,i,r} g_x^{uv} \{_u{}^r{}_i\}_x p_r p^i(x) p_v.$$

The right-hand side of this formula is a homogeneous polynomial of third degree in $p_1, p_2, \ldots, p_n$ with coefficients depending on x. The vanishing of the mean curvature $H_{p,x}$, identically with respect to p and x, implies the equations

$$g^{uv} \{_u{}^{(r}{}_v\} g^{st)} = g^{u(s} \{_u{}^r{}_v\} g^{t)v}, \qquad (\mathcal{E}^n)$$

where parentheses denote symmetrisation with respect to the included indices. With the notation

$$[ij, k] := g_{kr} \{_i{}^r{}_j\}$$

for the Christoffel symbols of the first kind of g and $f_{,i} = \partial f / \partial x^i$ for the partial derivatives, the covariant version of the system $(\mathcal{E}^n)$ becomes

$$g^{uv}([uv,r]g_{st} + [uv,s]g_{tr} + [uv,t]g_{rs}) = [(rs,t)] = \tfrac{1}{2}g_{(rs,t)}. \qquad (\mathcal{E}_n)$$

Theorem 1. *A riemannian structure (D,g), $D \subset \mathbb{R}^n$ is a solution of the problem $\mathcal{L}_n$ if and only if the components and the Christoffel symbols of the tensor g satisfy the system $\mathcal{E}^n$ or $\mathcal{E}_n$.*

3. The integration of the system $\mathcal{E}_n$

First recall the formula

$$g^{rs}[ir,s] = (\ln|g|^{1/2}),_i ,$$

where $|g| := \det(g_{ij})$, which is well known in riemannian geometry, see [5] for example. Contracting the equations $\mathcal{E}_n$ with g^{rs} one obtains

$$(n+2)g^{uv}[uv,t] = 2(\ln|g|^{1/2}),_t + g^{rs}[rs,t]$$

and therefore

$$(n+1)g^{uv}[uv,t] = (\ln|g|),_t .$$

Replacing in $\mathcal{E}_n$, this one writes

$$(\ln|g|),_{(r}\, g_{st)} = (n+1)[(rs,t)]$$

and also

$$(\ln|g|^{1/(n+1)}),_{(r}\, g_{st)} = \tfrac{1}{2}g_{(rs,t)}. \qquad (\mathcal{E}'_n)$$

At this point it seems natural to introduce a new riemannian structure G, conformally related to g, through

$$G_{ij} := |g|^{-2/(n+1)}g_{ij}.$$

One has

$$G_{ij,k} = |g|^{-2/(n+1)}\left(g_{ij,k} - \frac{2}{n+1}\frac{|g|,_k}{|g|}g_{ij}\right)$$

and system $\mathcal{E}'_n$ becomes

$$G_{(ij,k)} = 0. \qquad (\mathcal{E}''_n)$$

Moreover

$$|G| := \det(G_{ij}) = (|g|^{-2/(n+1)})^n|g| = |g|^{(1-n)/(n+1)}$$

and

$$|g| = |G|^{(n+1)/(1-n)}.$$

One recognizes in $\mathcal{E}''_n$ the equation which defines the second-order Killing tensor fields of the euclidean space $(\mathbb{R}^n, g_{\mathrm{E}})$. Every solution G of system $\mathcal{E}''_n$ provides a solution g of the system $\mathcal{E}_n$ given by

$$g_{ij} = |G|^{2/(1-n)} G_{ij},$$

which is defined in the domains D where $|G| \neq 0$. For the literature concerning Killing tensor fields the reader is referred to the article [3]. Killing tensor fields of the euclidean space are reducible; this means that all of them are linear combinations with constant coefficients of symmetric products of Killing vector fields of $(\mathbb{R}^n, g_{\mathrm{E}})$. In other words, the symmetric bilinear form $\Sigma G_{ij}\, \mathrm{d}X^i \circ \mathrm{d}X^j$ can be expressed in terms of the $n(n+1)/2$ Killing 1-forms of $(\mathbb{R}^n, g_{\mathrm{E}})$

$$\mathrm{d}X^i, X^i \wedge \mathrm{d}X^j := X^i\, \mathrm{d}X^j - X^j\, \mathrm{d}X^i, \quad i, j = 1, 2, ..., n$$

through the formula

$$\begin{aligned}
\sum_{i\le j} G_{ij}\, \mathrm{d}X^i \circ \mathrm{d}X^j = {} & \sum_{i\le j} a_{ij}\, \mathrm{d}X^i \circ \mathrm{d}X^j + \sum_{i\ne j} b_{ij}\, \mathrm{d}X^i \circ (X^i \wedge \mathrm{d}X^j) \\
& + \sum_{i\ne j;\ i,j<k} c_{ijk}\, \mathrm{d}X^i \circ (X^j \wedge \mathrm{d}X^k) \\
& + \sum_{i<j} d_{ij} (X^i \wedge \mathrm{d}X^j) \circ (X^i \wedge \mathrm{d}X^j) \\
& + \sum_{j<k;\ i\ne j,k} e_{ijk} (X^j \wedge \mathrm{d}X^i) \circ (X^k \wedge \mathrm{d}X^i) \\
& + \sum_{i<j<k<l} [h_{ij,kl} (X^i \wedge \mathrm{d}X^j) \circ (X^k \wedge \mathrm{d}X^l) \\
& + h_{ik,jl} (X^i \wedge \mathrm{d}X^k) \circ (X^j \wedge \mathrm{d}X^l)],
\end{aligned}$$

where $a_{ij}, b_{ij}, c_{ijk}, d_{ij}, e_{ijk}, h_{ij,kl}$ are

$$M_2(n) = n(n+1)^2(n+2)/12$$

arbitrary constants and $\circ$ denotes the symmetric tensor product

$$\mathrm{d}X^i \circ \mathrm{d}X^j = \tfrac{1}{2}(\mathrm{d}X^i \otimes \mathrm{d}X^j + \mathrm{d}X^j \otimes \mathrm{d}X^i).$$

Theorem 2. *The solutions g of the problem $\mathcal{L}_n$ come from the positive definite second-order Killing tensor fields G of the euclidean space through multiplication by a certain power ($|G|^{2/(1-n)}$) of the determinant $|G|$. The polynomial solutions of the problem $\mathcal{L}_n$ correspond to the Killing tensors G of constant determinant $|G|$. For $n = 3$, the solutions are always rational.*

4. A characterisation of the metrics $g_{\mathrm{H},k}$

The riemannian metrics

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + [\mathrm{d}z - \tfrac{1}{2}k(x\,\mathrm{d}y - y\,\mathrm{d}x)]^2, \qquad k \in \mathbb{R}^*$$

denoted $g_{\mathrm{H},k}$, represent polynomial solutions of the problem $\mathcal{L}_3$ in $\mathbb{R}^3$; their determinant is constant, $|g| = 1$. The riemannian metrics $g_{\mathrm{H},k}$ are invariant under left translations on the Lie group $(\mathbb{R}^3, \mu_k)$ defined in the introduction because the 1-forms $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z - \frac{1}{2}kx\,\mathrm{d}y + \frac{1}{2}ky\,\mathrm{d}x$ are so.

The group $(\mathbb{R}^3, \mu_k)$ is isomorphic with the Heisenberg group

$$H_1 = \left\{ \begin{pmatrix} 1 & X & Z \\ 0 & 1 & Y \\ 0 & 0 & 1 \end{pmatrix}, \; X, Y, Z \in \mathbb{R} \right\}.$$

The formulas

$$X = x, \qquad Y = y, \qquad Z = k^{-1}z + \tfrac{1}{2}xy$$

establish such an isomorphism and transform the metric $g_{\mathrm{H},k}$ into

$$\mathrm{d}\sigma_k^2 = \mathrm{d}X^2 + \mathrm{d}Y^2 + k^2(\mathrm{d}Z - X\,\mathrm{d}Y)^2$$

on H_1. One knows , see [4], that every left-invariant riemannian metric on H_1 is equivalent with some $\mathrm{d}\sigma_k^2$. We prove now that for $n = 3$, the condition $|G| =$ const. characterizes the metrics $g_{\mathrm{H},k}$; equivalently, the metrics $g_{\mathrm{H},k}$ are the only polynomial solutions of the problem $\mathcal{L}_3$ (modulo automorphisms of the euclidean space). The solution G of the system $\mathcal{E}_3''$ in a form which is more convenient to computations, after orthonormalisation of the basis of $\mathbb{R}^3$ with respect to $G(0)$, writes

$$\begin{aligned}\sum_{i \le j} G_{ij}\,\mathrm{d}X^i \circ \mathrm{d}X^j = {} & \mathrm{d}x^2 + \beta_{12}\,\mathrm{d}x \circ (y \wedge \mathrm{d}x) + \beta_{21}\,\mathrm{d}y \circ (x \wedge \mathrm{d}y) \\ & -\gamma_1 x\,\mathrm{d}y \circ \mathrm{d}z + \delta_1 (y \wedge \mathrm{d}z) \circ (y \wedge \mathrm{d}z) \\ & +\varepsilon_1 (x \wedge \mathrm{d}y) \circ (x \wedge \mathrm{d}z) + \cdots,\end{aligned}$$

where points stand for terms one can obtain from those already written under cyclic permutations of the letters x, y, z and the indices 1, 2, 3. The 15 constants $\beta_{ij}, \gamma_i, \varepsilon_i$ are related by

$$\gamma_1 + \gamma_2 + \gamma_3 = 0.$$

The coefficients of the quadratic form G are

$$G_{11} = 1 + \beta_{12}y + \beta_{13}z + \delta_3 y^2 + \delta_2 z^2 + \varepsilon_1 yz,$$

$$-G_{12} = \beta_{12}x + \beta_{21}y + \gamma_3 z + 2\delta_3 xy + \varepsilon_1 xz + \varepsilon_2 yz - \varepsilon_3 z^2,$$

and those obtained from them under cyclic permutations. By direct calculations, in order that $|G| = 1$, one finds the following conditions:

(i) $\beta_{21} + \beta_{31} = 0, \ldots,$
(ii) $2\delta_1 = -\frac{1}{2}\gamma_2\gamma_3 - \frac{1}{2}\beta_{21}^2 + \beta_{12}^2 + \beta_{13}^2, \ldots,$
(iii) $\varepsilon_1 = \frac{3}{2}\beta_{32}\beta_{23} + \frac{1}{2}\beta_{21}\gamma_3 + \frac{1}{2}\beta_{31}\gamma_2, \ldots,$
(iv) $\gamma_1\beta_{12}\beta_{13} + \beta_{31}(\beta_{32}^2 - \beta_{23}^2) = 0, \ldots,$
(v) $\gamma_1\gamma_2\gamma_3 + \gamma_1\beta_{31}^2 + \gamma_2\beta_{12}^2 + \gamma_3\beta_{23}^2 = 0,$
(vi) $\beta_{12}\beta_{31}(\gamma_1 - \gamma_3) + \beta_{23}(\gamma_1\gamma_2 + \beta_{31}^2 - 2\beta_{32}^2 + \beta_{23}^2) = 0, \ldots .$

The proof will be done in three steps.

Step 1: Suppose $\beta_{12}\beta_{23}\beta_{31} \neq 0$. The precedent algebraic system defines all the parameters in terms of $\beta_{12}, \beta_{23}, \beta_{31}$ and the quadratic form corresponding to G becomes

$$\begin{aligned} \mathrm{d}S^2 &:= \sum_{i\leq j} G_{ij}\,\mathrm{d}X^i\mathrm{d}X^j \\ &= (\beta^4/4)\omega^2 + \beta^{-4}[\beta_{31}^2(\beta_{12}^2 + \beta_{23}^2)\,\mathrm{d}x^2 + \cdots + 2\beta_{23}^2\beta_{31}\beta_{32}\,\mathrm{d}x\,\mathrm{d}y + \cdots], \end{aligned}$$

where

$$\begin{aligned} \omega &:= \beta_{23}^{-1}(y\,\mathrm{d}x - x\,\mathrm{d}y) + \beta_{31}^{-1}(z\,\mathrm{d}y - y\,\mathrm{d}z) + \beta_{12}^{-1}(x\,\mathrm{d}z - z\,\mathrm{d}x) \\ &\quad + 2\beta^{-4}(\beta_{23}\beta_{12}\,\mathrm{d}x + \beta_{31}\beta_{23}\,\mathrm{d}y + \beta_{12}\beta_{31}\,\mathrm{d}z) \end{aligned}$$

and

$$\beta^4 := \beta_{12}^2\beta_{23}^2 + \beta_{23}^2\beta_{31}^2 + \beta_{31}^2\beta_{12}^2.$$

With respect to a new orthonormal frame, chosen so that the new coordinate z becomes

$$Z := \beta^{-2}(\beta_{23}\beta_{12}x + \beta_{31}\beta_{23}y + \beta_{12}\beta_{31}z),$$

one finds that

$$(\beta^2/2)\omega = \mathrm{d}Z + (\beta^4/(2\beta_{12}\beta_{23}\beta_{31}))[Y\,\mathrm{d}X - X\,\mathrm{d}Y].$$

The quadratic form

$$\beta^{-4}[\beta_{31}^2(\beta_{12}^2 + \beta_{23}^2)\,\mathrm{d}x^2 + \cdots + 2\beta_{23}^2\beta_{31}\beta_{32}\,\mathrm{d}x\,\mathrm{d}y + \cdots],$$

which is of rank 2, invariant under rotations about the OZ-axis and vanishes in the OZ-direction, becomes $\mathrm{d}X^2 + \mathrm{d}Y^2$ with respect to the new orthonormal coordinates X, Y, Z. Therefore, with respect to the new coordinates,

$$\mathrm{d}S^2 = [\mathrm{d}Z + (\beta^4/(2\beta_{12}\beta_{23}\beta_{31}))(Y\,\mathrm{d}X - X\,\mathrm{d}Y)]^2 + \mathrm{d}X^2 + \mathrm{d}Y^2,$$

which is a riemannian metric of $g_{\mathrm{H},k}$ type.

Step 2: Suppose now $\beta_{12}\beta_{23}\beta_{31} = 0$ but $\beta_{23} \neq 0$. Eq. (iv) shows that $\beta_{12} = \beta_{31} = 0$ but from (vi) one deduces that

$$\gamma_1\gamma_2 = -\beta_{23}^2 \neq 0.$$

Due to this fact ($\gamma_1\gamma_2 \neq 0$) one can rotate the frame about the Ox-axis in order to get a new frame $Ox'y'z'$ with respect to which $\beta'_{12}\beta'_{23}\beta'_{31} \neq 0$.

Step 3: If all the β_{ij} vanish, Eq. (v) gives:

$$\gamma_1\gamma_2\gamma_3 = 0$$

and at least one of the γ_i's vanish, say $\gamma_3 = 0$. Then, $\gamma_2 = -\gamma_1$ and one has

$$\mathrm{d}S^2 = [\mathrm{d}z + (\gamma_2/2)(x\,\mathrm{d}y - y\,\mathrm{d}x)]^2 + \mathrm{d}x^2 + \mathrm{d}y^2.$$

Theorem 3. *Modulo an isometry of the euclidean space* ($\mathbb{R}^3, g_E$), *the riemannian metrics* $g_{H,k}$ *are the only polynomial solutions of the* $\mathcal{L}_3$ *problem.*

It would be suitable to know the polynomial solutions of the $\mathcal{L}_n$ problem for $n > 3$.

5. Final remarks

Remark 1. The problem $\mathcal{L}_n$ can be formulated for pseudo-riemannian metrics too and Theorems 1 and 2 get unchanged.

Remark 2. One can slightly modify the problem into say $\mathcal{L}_n^c$ by looking for the riemannian metrics g, defined in domains D of $\mathbb{R}^n$, for which all the linear varieties of codimension c are minimal. Problem $\mathcal{L}_n^1$ coincides with $\mathcal{L}_n$ but for $c > 1$ the problem $\mathcal{L}_n^c$ is not interesting in the sense that its solutions are nothing but constant curvature metrics.

Remark 3. Projective diffeomorphisms permute solutions of the $\mathcal{L}_n$ problem. This means that system $\mathcal{E}_n$ is projectively invariant fact which is not evident in its present form. But one can transform it remembering that for projectively related riemannian metrics g and g' i.e. riemannian metrics which admit the same geodesics, the Christoffel symbols $\{{}_j{}^i{}_k\}$ and $\{{}_j{}^i{}_k\}'$ are related by the formulas, see [5],

$$\{{}_j{}^i{}_k\}' = \{{}_j{}^i{}_k\} + \delta^i_j\varphi_k + \delta^i_k\varphi_j.$$

Contracting for $i = j$ one gets

$$(\ln|g'|^{1/2})_{,k} = (\ln|g|^{1/2})_{,k} + (n+1)\varphi_k.$$

Eliminating the φ_k's between these equations, one gets a projective tensor field $P(g)$, with components

$$P^i_{jk} := \{{}_j{}^i{}_k\} - \frac{1}{n+1}\delta^i_j(\ln|g|^{1/2})_{,k} - \frac{1}{n+1}\delta^i_k(\ln|g|^{1/2})_{,j}\ ,$$

which is invariant under projective diffeomorphisms. With the notation

$$P_{jk\,i} := \sum_r P^r_{jk}g_{ri} = [jk,i] - \frac{1}{n+1}g_{ij}(\ln|g|^{1/2})_{,k} - \frac{1}{n+1}g_{ik}(\ln|g|^{1/2})_{,j}$$

for its covariant components, the system $\mathcal{E}_n$ becomes

$$\sum_{uv} g^{uv} P_{uv(r}g_{st)} = P_{(rs\ t)} \qquad (\mathcal{E}_n^*)$$

and now the projective character of the system is evident.

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